

ON THE UNRAMIFIED BRAUER GROUP OF A HOMOGENEOUS SPACE

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ABSTRACT. We give a new proof of the theorem stating that for any connected linear algebraic group G over an algebraically closed field k of characteristic 0 and for any connected closed subgroup H of G , the unramified Brauer group of G/H vanishes.

1. INTRODUCTION

In this note k always denotes an algebraically closed field of characteristic 0. For an irreducible algebraic variety X over k , we denote by $k(X)$ the field of rational functions on X . We denote by $\mathrm{Br}_{\mathrm{nr}} k(X)$, or just by $\mathrm{Br}_{\mathrm{nr}} X$, the unramified Brauer group of $k(X)$ with respect to k , see [CTS, Def. 5.3].

We give a new proof of the following theorem:

Theorem 1 ([BDH, Thm. 5.1]). *Let G be a connected linear algebraic group over an algebraically closed field k of characteristic 0, and let $H \subset G$ be a connected closed subgroup. Then $\mathrm{Br}_{\mathrm{nr}} k(G/H) = 0$.*

In the case when G is simply connected this is a classical result of Bogomolov [Bog, Thm. 2.4], see Colliot-Thélène and Sansuc [CTS, Thm. 9.13]. Bogomolov proved his theorem by a topological method in the case $k = \mathbb{C}$, but the general case of an algebraically closed field k of characteristic 0 reduces to the case $k = \mathbb{C}$ by the Lefschetz principle, see [CTS], beginning of § 9. Our Theorem 1 answers affirmatively a question of Colliot-Thélène and Sansuc in [CTS, Rem. 9.14] and a question after Theorem 1.4 in the paper [CTK] by Colliot-Thélène and Kunyavskiĭ. Theorem 1 was recently proved in the preprint [BDH] of Demarche, Harari and the author by a number-theoretical method. Here we deduce this theorem from Bogomolov's theorem.

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2. NOTATION AND PRELIMINARIES

By k we always denote an algebraically closed field of characteristic 0. Let G be a connected linear algebraic group over k . We use the following notation:

G^u is the unipotent radical of G ;

$G^{\text{red}} = G/G^u$, it is reductive;

$G^{\text{ss}} = [G^{\text{red}}, G^{\text{red}}]$, it is semisimple;

$G^{\text{tor}} = G^{\text{red}}/G^{\text{ss}}$, it is a torus;

$G^{\text{ssu}} = \ker[G \rightarrow G^{\text{tor}}]$, it is an extension of a semisimple group G^{ss} by a unipotent group G^u .

Note that G^{tor} is the largest quotient torus of G . Note also that $\text{Pic } G = 0$ if and only if G^{ssu} is simply connected, cf. [Sa], Lemma 6.9 and Remark 6.13.

Let X be a smooth integral variety over k . If V is a smooth compactification of X (existing by Hironaka's theorem), then we can identify $\text{Br}_{\text{nr}} X$ with $\text{Br } V$, cf. [CTS, Thm. 5.11(iii)]. We regard $\text{Br}_{\text{nr}} X = \text{Br } V$ as a subgroup of $\text{Br } X$, cf. [CTS, Thm. 5.11(ii)]. If $f: X_1 \rightarrow X_2$ is a morphism of smooth integral varieties defined over k , one can extend f to a morphism of suitable smooth compactifications $f': V_1 \rightarrow V_2$, where V_i is a smooth compactification of X_i ($i = 1, 2$), see [BK, § 1.2.2] (again, one uses Hironaka's theorem). It follows that f induces a homomorphism of the unramified Brauer groups $f^{\text{nr}}: \text{Br}_{\text{nr}} X_2 \rightarrow \text{Br}_{\text{nr}} X_1$ fitting into a commutative diagram

$$(1) \quad \begin{array}{ccc} \text{Br}_{\text{nr}} X_2 & \xrightarrow{f^{\text{nr}}} & \text{Br}_{\text{nr}} X_1 \\ \downarrow & & \downarrow \\ \text{Br } X_2 & \xrightarrow{f^*} & \text{Br } X_1. \end{array}$$

3. REDUCTION TO A SPECIAL CASE

Let G be a connected linear algebraic group defined over k , and let $H \subset G$ be a connected closed subgroup.

3.1. Reduction to the case $\text{Pic } G = 0$. It is well known (see e.g. [Bor, Lemma 5.2]) that there exists a connected linear algebraic group G' over k with $\text{Pic } G' = 0$ and a connected closed subgroup $H' \subset G'$, such that the varieties G/H and G'/H' are isomorphic. Therefore, we may and shall assume that G in Theorem 1 satisfies $\text{Pic } G = 0$.

3.2. Reduction to the case $H = H^{\text{ssu}}$. Consider the subgroup $H^{\text{ssu}} \subset H$. The map $G/H^{\text{ssu}} \rightarrow G/H$ is a right H^{tor} -torsor. Since H^{tor} is a split torus, by Hilbert's Theorem 90 this torsor admits a local section. Thus the homogeneous space G/H^{ssu} is birationally equivalent to $G/H \times_k H^{\text{tor}}$, and by [CTS, Prop. 5.7] we have $\text{Br}_{\text{nr}}(G/H^{\text{ssu}}) \cong \text{Br}_{\text{nr}}(G/H)$. Therefore, we may and shall assume in Theorem 1 that $H = H^{\text{ssu}}$, i.e. H is character-free.

4. DEDUCTION OF THEOREM 1 FROM BOGOMOLOV'S THEOREM

Consider the map $G \rightarrow G/H$. Since the variety of G is rational, we have $\mathrm{Br}_{\mathrm{nr}} G = 0$, and we see from diagram (1) that

$$(2) \quad \mathrm{Br}_{\mathrm{nr}}(G/H) \subset \ker[\mathrm{Br}(G/H) \rightarrow \mathrm{Br} G].$$

Taking in account the results of Section 3, we now assume that $\mathrm{Pic} G = 0$ and that $H \subset G$ is connected and character-free. Set $G_1 = G^{\mathrm{ssu}}$, then G_1 is simply connected because $\mathrm{Pic} G = 0$, see § 2. Since H is character-free, we have $H \subset G_1$.

Let $i: G_1 \hookrightarrow G$ denote the inclusion homomorphism. Consider the following commutative diagram of morphisms of varieties:

$$(3) \quad \begin{array}{ccc} G_1 & \xrightarrow{i} & G \\ \downarrow & & \downarrow \\ G_1/H & \xrightarrow{i_*} & G/H. \end{array}$$

By functoriality (see § 2) this diagram defines a homomorphism $i^{\mathrm{nr}}: \mathrm{Br}_{\mathrm{nr}}(G/H) \rightarrow \mathrm{Br}_{\mathrm{nr}}(G_1/H)$ fitting into a commutative diagram

$$(4) \quad \begin{array}{ccc} \mathrm{Br}_{\mathrm{nr}}(G/H) & \xrightarrow{i^{\mathrm{nr}}} & \mathrm{Br}_{\mathrm{nr}}(G_1/H) \\ \downarrow & & \downarrow \\ \mathrm{Br}(G/H) & \xrightarrow{i^*} & \mathrm{Br}(G_1/H). \end{array}$$

Note that the map $i: G_1 \rightarrow G$ in diagram (3) is an H -equivariant map from the right H -torsor G_1 over G_1/H to the right H -torsor G over G/H . Sansuc's exact sequence [Sa, (6.10.1)], applied to this diagram, gives a commutative diagram with exact rows

$$(5) \quad \begin{array}{ccccccc} 0 = \mathrm{Pic} G & \longrightarrow & \mathrm{Pic} H & \longrightarrow & \mathrm{Br}(G/H) & \longrightarrow & \mathrm{Br} G \\ & & \downarrow \mathrm{id} & & \downarrow i^* & & \downarrow \\ 0 = \mathrm{Pic} G_1 & \longrightarrow & \mathrm{Pic} H & \longrightarrow & \mathrm{Br}(G_1/H) & \longrightarrow & \mathrm{Br} G_1. \end{array}$$

We see from (5) that the homomorphism i^* restricted to $\ker[\mathrm{Br}(G/H) \rightarrow \mathrm{Br} G]$ is injective, and we see from (2) that i^* restricted to $\mathrm{Br}_{\mathrm{nr}}(G/H)$ is injective. Now it follows from diagram (4) that the homomorphism

$$i^{\mathrm{nr}}: \mathrm{Br}_{\mathrm{nr}}(G/H) \rightarrow \mathrm{Br}_{\mathrm{nr}}(G_1/H)$$

is injective. Since G_1 is simply connected and H is connected, by Bogomolov's theorem [CTS, Thm. 9.13] we have $\mathrm{Br}_{\mathrm{nr}}(G_1/H) = 0$. We conclude that $\mathrm{Br}_{\mathrm{nr}}(G/H) = 0$, which proves Theorem 1. \square

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